Does Charge Contribute to the Frame Dragging of Spacetime?

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Abstract Electrically charged systems bound by a strong gravitational force can sustain a huge amount of electric charge (up to 10^{20} C) against Coulomb repulsion. General relativistically such systems form a stable hydrostatic configuration both in the non-rotating and rotating cases. Here we study the effects of electric charge (electric energy density) on the spacetime outside a rotating electrically charged system bound by a strong gravitational force. In particular we investigate the effect of charge density on frame-dragging of spacetime in the exterior region. Using the coupled Einstein-Maxwell equations it is found that in the slow rotation approximation charge accumulation not only acts like an additional mass, thus modifying the spherically symmetric part of the spacetime, the electric charge also contributes directly to the dragging of spacetime. A modified Lense-Thirring formula for the spacetime frame dragging frequency is obtained and its implication for rotating charged compact stars is discussed.

1 Introduction

Hydrostatic equilibrium in physical systems bound by Newtonian gravitation is achieved essentially under electrically neutral conditions [1, 2]. A comparison between the repulsive Coulomb force and the attractive gravitational force shows that for systems such as the Sun, charge accumulation cannot be significantly larger than 10^2 C. For systems with intense gravitational field such as compact stars the Newtonian theory does not lead to correct equilibrium conditions, here modifications in the gravitational field equations consistent with the general theory of relativity must be introduced. This leads to the general relativistic hydrostatic conditions (the Tolman-Oppenheimer-Volkoff or TOV equations) for equilibrium in relativistic stars. The relativistic hydrostatic equilibrium conditions for electrically charged

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systems imply that charge contained by these systems can be very large. It is estimated that both in non-rotating [3–9] as well as rotating cases [10] the amount of net charge contained in a compact charged star can be as high as 10^{20} C. In case of a non-rotating charged compact star (electar) the spacetime is modified by the electrical energy density. Here the effect of charge appears as an addition to the total mass of the star (system), hence the its energy density. For rotating gravitational systems frame-dragging of spacetime is a typically relativistic effect with no analogue in classical theories of gravitation [11]. In the case of a neutral massive object, a neutron star for instance, the spacetime dragging effects both material as well as radiative processes in vicinity of the star.

In this paper we study the effect of charge on the spacetime outside a rotating electrically charged relativistic system, such as a charged compact star. In particular we ask if the dragging of spacetime influenced by the presence of electric charge on a relativistic star. In Sect. 2 the problem is formulated assuming a slowly rotating mass with an axially symmetric metric describing the spacetime around the gravitational source. The form of the metric is then determined using Einstein field equations coupled to the Maxwell equations. This leads to a modified formula for Lense-Thirring spacetime dragging frequency, given in Sect. 3. In Sect. 4 we discuss the main conclusions and give a summary of the paper.

2 General Relativistic Formulation

We consider an electrically charged, slowly rotating star of total mass M and radius R enclosing a net electric charge Q. The spacetime exterior to the star is described by the metric

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - 2\omega(r)r^2 \sin^2\theta dt d\varphi - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (1)$$

where $\omega(r)$ is the dragging frequency of a local inertial frame at the location r outside the star. The spacetime exterior to the charged star will be explicitly determined using the coupled Einstein-Maxwell equations,

$$R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R = -8\pi T^\alpha_\beta, \qquad (2)$$

supplemented with the stress energy-momentum tensor for the electromagnetic field given by

$$T^\alpha_\beta = \frac{1}{4\pi}\left(F^{\alpha\gamma}F_{\gamma\beta} - \frac{1}{4}\delta^\alpha_\beta F_{\mu\nu}F^{\mu\nu}\right), \qquad (3)$$

where the electromagnetic field tensor $F^{\mu\nu}$ is given by the Maxwell equations:

$$(\sqrt{-g}F^{\alpha\beta})_{,\beta} = 4\pi\sqrt{-g}J^\alpha. \qquad (4)$$

For electric charge density $\rho_e(r)$ we define the electric field $E(r)$ as

$$E(r) = \frac{4\pi}{r^2}\int_0^r \rho_e(r')r'^2 e^{\lambda(r')/2}dr', \qquad (5)$$

then for the components T^t_t and T^r_r the stress tensor can be expressed as

$$\frac{1}{4\pi}\left(F^{\alpha\gamma}F_{\gamma\beta} - \frac{1}{4}\delta^\alpha_\beta F_{\mu\nu}F^{\mu\nu}\right) = -\frac{E^2}{8\pi}; \quad \alpha = \beta = 0, 1, \qquad (6)$$

and $E^2/8\pi = E(R)^2/8\pi$ is the electric energy density of the star. For $r > R$ the spherical symmetric parts of the Einstein field equation coupled to the Maxwell equation give

$$\frac{e^{-\lambda}}{r^2}\left(r\frac{d\lambda}{dr} - 1\right) + \frac{1}{r^2} = E^2, \tag{7}$$

$$\frac{e^{-\lambda}}{r^2}\left(r\frac{d\nu}{dr} + 1\right) - \frac{1}{r^2} = -E^2. \tag{8}$$

The first of Einstein field equations gives the metric component $e^{-\lambda(r)}$ as

$$e^{-\lambda(r)} = 1 - \frac{8\pi}{r}\int_0^r \rho r^2 dr = 1 - \frac{2M(r)}{r}, \quad r > R \tag{9}$$

where

$$M(r) = 4\pi\int_0^r \left(\rho(r) + \frac{E^2}{8\pi}\right)r^2 dr, \tag{10}$$

is the modified mass. Also adding these two equations together and solving the resulting expression we get $\nu(r) = -\lambda(r)$. Corresponding to the axially symmetric part of the Einstein field equations $G_\varphi^t = 8\pi T_\varphi^t$ we obtain the following equation

$$e^{-\lambda(r)}\left(r\frac{d^2\omega}{dr^2} + 4\frac{d\omega}{dr}\right) = 16\pi r E^2\omega. \tag{11}$$

In terms of the net charge $Q = 4\pi\int_0^R \rho_e r^2 e^{\lambda(r)/2}dr$ the last equation can be expressed as

$$\frac{d}{dr}\left(r^4\frac{d\omega(r)}{dr}\right) = 16\pi Q^2\omega(r)e^{\lambda(r)} \tag{12}$$

where $e^{\lambda(r)} = r/(r - 2M)$. In this equation the dragging frequency outside the star is modified by the electrical energy due the presence of a net electric charge in the star.

3 Modified Lense-Thirring Frame Dragging Frequency

Equation (12) cannot be solved exactly, however for $M/r \ll 1$ it is possible to obtain a physically interesting solution to (12), such as for the case of a charged compact star. Under this approximation equation (12) becomes

$$\frac{d^2\omega}{dr^2} + \frac{4}{r}\frac{d\omega}{dr} - \frac{q}{r^4}\omega = 0, \tag{13}$$

where we define $q \equiv 16\pi Q^2$. We notice that (13) has a regular singular point at r whereas (12) has a removable singularity at $r = 2M$ also. Solution to (13) can be obtained by standard integration procedure, which gives

$$\omega(r) = \frac{r - q}{r}C_1 e^{q/r} - \frac{r + q}{2q^3 r}C_2 e^{-q/r}, \quad q \neq 0, \tag{14}$$

where C_1 and C_2 are constants of integration. Equation (14) is only an approximate solution of the original equation (12). However a comparison between numerical solutions to (12)
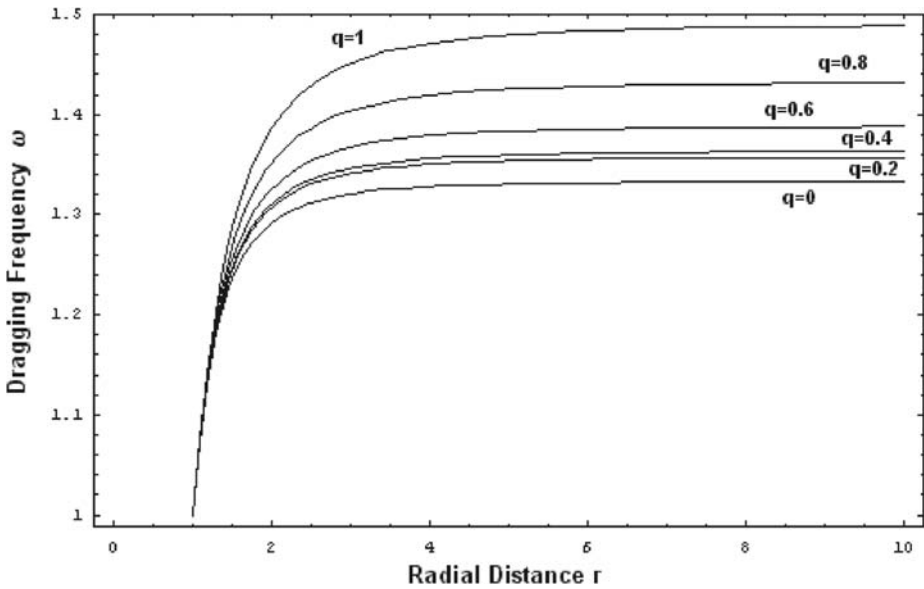
Fig. 1 Plots between the spacetime dragging frequency $\omega(r)$ and the radial distance r using (12) for different values of the net charge q

and solution (14) shows that solution matches correctly to the exact solution for $M/r < 1$. A drawback of the solution (14) seems to be its non-reducibility to the case of an uncharged electrically neutral object, that is the limit when $q \to 0$. To investigate this more fully we expand the exponentials in (14) in a power series, after some simplification we have

$$\omega(r) = C_1\left(1 - \frac{q^2}{2r^2} - \frac{q^3}{3r^3} - \cdots\right) - C_2\frac{1}{2q^3}\left(1 - \frac{q^2}{2r^2} + \frac{q^3}{3r^3} - + \cdots\right), \quad q \neq 0, \quad (15)$$

separating terms independent of q we obtain

$$\omega(r) = -\frac{C_2}{3r^3} + C_1 - \left\{C_1\left(\frac{q^2}{2r^2} + \frac{q^3}{3r^3} + \cdots\right) + C_2\left(\frac{1}{2q^3} - \frac{1}{4qr^2} + - \cdots\right)\right\}, \quad q \neq 0. \quad (16)$$

Here the first two terms give the correct solution to (14) for the chargeless case. For the remaining terms in (16) it can be seen that as q approaches zero, the terms in parentheses together become vanishingly small for a fixed value of r.

In Fig. 1 we give exact numerical solutions of (12) for different values of q. For comparison with the approximate formula (14) for modified Lense-Thirring frequency, plots are given in Fig. 2, between the dragging frequency $\omega(r)$ and the radial distance r. In both cases we take $\omega(1) = 1 = \omega'(r)$ and $M = 0.2$ in gravitational units $G = 1 = c$. Figure 3 consists of plots for the spherically symmetric part of the metric $e^{-\lambda(r)}$ as a function of distance r for various values of net charge q inside the star. We notice that the modified Lense-Thirring frequency has the same profile in each case for different values of charge q.
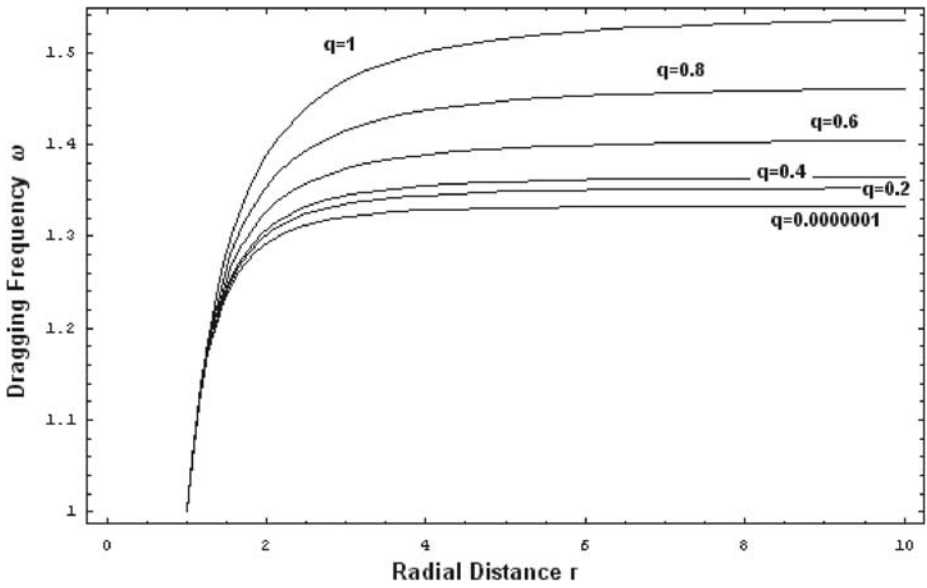
Fig. 2 Plots between the spacetime dragging frequency $\omega(r)$ and the radial distance r using the modified Lense-Thirring formula (14) for different values of the net charge q
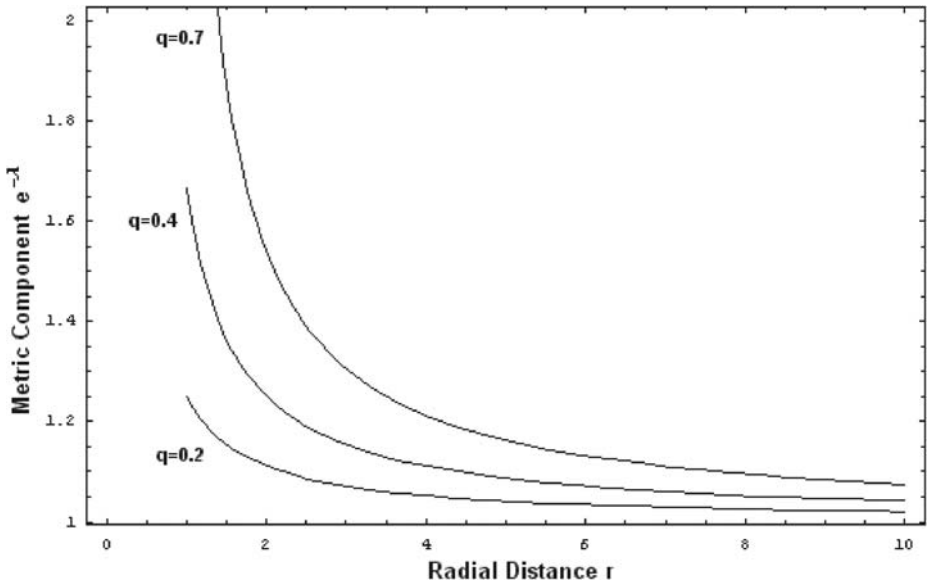

Fig. 3 Plots between the modified metric component $e^{-\lambda(r)}$ and the radial distance r using (9) for the exterior region with different values of the net charge q

4 Summary and Conclusions

We have studied the spacetime around a rotating electrically charged massive object within the context of the general theory of relativity. The spacetime metric is determined using the coupled set of Einstein-Maxwell field equations for the case of a rotating electrically charged system. These equations are then solved to obtain analytical expressions for the metric components.

The solutions obtained show that the presence of charge effects the spacetime in a twofold way. Firstly, it modifies the total energy density hence the total mass of the system. This in turn effects the spherically symmetric part of the spacetime metric (Fig. 3). Secondly, the axially symmetric part of the metric due to the rotation of the system modifies the Lense-Thirring spacetime dragging around the gravitating object (Figs. 1 and 2). We obtain an analytical formula for the modified Lense-Thirring dragging frequency which is applicable to the case of charged compact objects under the approximation $M/r \ll 1$. We find that charged massive object (such as a charged compact star) is not only more strongly gravitating due to the enhanced total mass (energy density), but if rotating the dragging of spacetime around the massive object is modified also.

This effect has important implications for electrically charged gravitationally bound stellar systems. For instance a free falling inertial observer (ZAMO) with velocity four vector $u_{\text{ZAMO}} = (e^{v(r)}, 0, 0, \omega(r)e^{v(r)})$ circling the star at fixed radial distance will not only need larger potential energy to keep itself in orbit, but also will revolve faster due to the enhanced dragging frequency [12]. Estimated charge contained in an electrically compact star can be as high as 10^{20} C. This corresponds to the mass increase of about ten to twenty percent mass of the star in a neutral condition [7–10]. Observationally the detection of such effects, for instance around a charged electric star (electar), seems therefore quite feasible.

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